PROJECTED WRITTEN NOTES FROM THE M325K LECTURE ON TUESDAY, MARCH 5, 2024, ON WRITING ELEMENTAL PROOFS IN SET THEORY

CLASS # 15

The Two Types of Proofs in Set Theory

Elemental Proofs

1. We work with individual elements in sets
2. The Justifications used are "By def'n of..." or "By Elimination, say" ← Standard Valid Argument.
3. You may not use Set Identities.

ALGEBAIC PROOFS

1. We work with whole sets at a time.
2. The Justifications used are "By Set Identity such-and-such"

Set Identities

An **identity** is an equation that is universally true for all elements in some set. For example, the equation $a + b = b + a$ is an identity for real numbers because it is true for all real numbers a and b. The collection of set properties in the next theorem consists entirely of set identities. That is, they are equations that are true for all sets in some universal set.

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

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The circled identities must be learned and applied by name.

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Recall the Definition, for sets A and B, of "$A \subseteq B$".

$A \subseteq B \iff \underbrace{\text{For every } x \in A,\ x \in B}$.

The definition is a universal statement often proved using the Direct Proof Method.

To Prove: For all sets X and Y, $X \subseteq (X \cup Y)$.

Proof: Let X and Y be any sets.
[NTS: $X \subseteq (X \cup Y)$]

Let t be any element of X.
[NTS: $x \in (X \cup Y)$]

$\therefore t \in X$.

$\therefore t \in X$ OR $t \in Y$, by generalization

$\therefore t \in X \cup Y$, by def'n of "Union".

$\boxed{\begin{array}{l}\therefore \text{For all } x \in X,\ x \in (X \cup Y), \text{ by direct proof.} \\ \therefore X \subseteq (X \cup Y), \text{ by def'n of "subset".}\end{array}}$

Proving $X \subseteq (X \cup Y)$ by the Direct Proof Method

$\boxed{\begin{array}{l}\therefore \text{FOR ALL SETS X and Y,} \\ X \subseteq (X \cup Y), \text{ by Direct Proof.}\end{array}}$

Proving the universal statement "For all sets..." by the Direct Proof Method.

QED

SEE THE NEXT PAGE FOR SHORTCUTS YOU CAN USE.

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The definition is a universal statement often proved using the Direct Proof Method.

To Prove: For all sets X and Y, $X \subseteq (X \cup Y)$.

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$\therefore t \in X$ OR $t \in Y$, by generalization.

$\therefore t \in X \cup Y$, by defin of "Union".

$\boxed{\therefore X \subseteq (X \cup Y), \text{ by Direct Proof.}}$

$\boxed{\text{QED, by Direct Proof.}}$

Proving the universal statement "For all sets.." by the Direct Proof Method.

Proving $X \subseteq (X \cup Y)$ by the Direct Proof Method

Guidelines for Writing Elemental Proofs in SET THEORY AND OTHER NOTES

Definition: A set A is non-empty or $A \neq \emptyset$

$\Leftrightarrow$ There exists an element $x \in \mathcal{U}$ such that $x \in A$.

Guidelines for Writing Elemental Proofs

① $\therefore x \in A \cap B$ [THIS IS FOLLOWED BY:]

$\therefore x \in A$ AND $x \in B$ by definition of "Intersection"

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② $\therefore x \in A \cup B$ [THIS IS FOLLOWED BY:]

$\therefore x \in A$ OR $x \in B$ by definition of "Union"

CASE 1 ($x \in A$)

... $\therefore$ ~ IN CASE 1.

CASE 2 ($x \in B$)

... $\therefore$ ~ IN CASE 2.

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③ $\therefore C \cap D \neq \emptyset$ [IS FOLLOWED BY:]

$\therefore$ There exists an element $x \in \mathcal{U}$ such that $x \in C \cap D$. [see ① Next]

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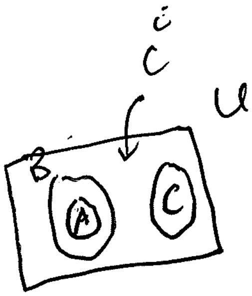
④ PROVE "$A \cap B = \emptyset$" using a Proof-by-Contradiction.

<u>FACT</u>: When you need to prove that a particular set is the EMPTY SET, ϕ, ALWAYS USE PROOF-BY-CONTRADICTION.

<u>Proposition 6.2.6</u>: For all sets A, B, C,

if $A \subseteq B$ and $B \subseteq C^c$, Then $A \cap C = \phi$.



<u>Proof</u>: Let A, B, C be any sets.

Suppose $A \subseteq B$ and $B \subseteq C^c$. [NTS: $A \cap C = \phi$].

Suppose, BWOC, that $A \cap C \neq \phi$.

$\therefore$ By def'n of "non-empty set", there exists an element $x \in U$ such that $x \in (A \cap C)$.

$\therefore x \in A$ and $x \in C$, by def'n of "intersection".

Since $A \subseteq B$, and $x \in A$, $x \in B$, by def'n of "subset".

Since $x \in B$ and $B \subseteq C^c$, $x \in C^c$ by def'n of "subset".

$\therefore x \notin C$ by def'n of set complement.

$\therefore x \in C$ and $x \notin C$, a contradiction.

$\therefore A \cap C = \phi$, by proof-by-contradiction.

$\therefore$ For all sets A, B, C,

if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \phi$,

by Direct Proof. QED.

EXAMPLE ELEMENTAL PROOF

Let $A = \{n \in \mathbb{Z}$ such that $12 \mid n\}$

$B = \{n \in \mathbb{Z}$ such that $15 \mid n\}$

$C = \{n \in \mathbb{Z}$ such that $60 \mid n\}$

<u>To Prove</u>: $C \subseteq (A \cap B)$.

Proof: [NTS: $C \subseteq (A \cap B)$]

["Let $n \in C$ be given" means "let n be any element of C"]

Let $n \in C$ be given. [NTS: $n \in (A \cap B)$]

Since $n \in C$, $60 \mid n$, by def'n of set C.

$\therefore$ There exists an integer t such that $n = 60t$.

[Showing $n \in B$] $n = 15(4t)$ and $4t$ is an integer.

$\therefore 15 \mid n$, and so, $n \in B$, by def'n of set B.

[Showing $n \in A$] $n = 12(5t)$ and $5t$ is an integer

$\therefore 12 \mid n$, and so, $n \in A$.

$\therefore n \in A$ and $n \in B$, by conjunction.

$\therefore n \in (A \cap B)$ by def'n of "$\cap$".

$\therefore C \subseteq (A \cap B)$, by Direct Proof.

QED.

To Prove: For all sets A and B,

if $A \subseteq B$, then $B^c \subseteq A^c$.

Proof: Let A and B be any sets.

Suppose $A \subseteq B$. [NTS: $B^c \subseteq A^c$].

Let $x \in B^c$ be given. [NTS: $x \in A^c$]

$\therefore x \notin B$, by def'n of B^c [NTS: $x \notin A$].

Suppose, BWOC, that $x \in A$.

Since $A \subseteq B$ and $x \in A$, $x \in B$, by def'n of "subset"

$\therefore x \in B$ and $x \notin B$, a contradiction.

$\therefore x \notin A$, by proof-by-contradiction.

$\therefore x \in A^c$, by def'n of A^c.

$\therefore B^c \subseteq A^c$, by direct proof.

$\therefore$ For all $x \in B^c$, $x \in A^c$ by Direct proof
$\therefore B^c \subseteq A^c$ by def'n of "subset"

$\therefore$ For all sets A and B,

if $A \subseteq B$, then $B^c \subseteq A^c$

by Direct Proof.

QED.